

SUBSETS OF VERTICES GIVE MORITA EQUIVALENCES OF LEAVITT PATH ALGEBRAS

LISA ORLOFF CLARK, ASTRID AN HUEF, AND PAREORANGA LUITEN-APIRANA

ABSTRACT. We show that every subset of vertices of a directed graph E gives a Morita equivalence between a subalgebra and an ideal of the associated Leavitt path algebra. We use this observation to prove an algebraic version of a theorem of Crisp and Gow: certain subgraphs of E can be contracted to a new graph G such that the Leavitt path algebras of E and G are Morita equivalent. We provide examples to illustrate how desingularising a graph, and in- or out-delaying of a graph, all fit into this setting.

1. INTRODUCTION

Given a directed graph E , Crisp and Gow identified in [10, Theorem 3.1] a type of subgraph which can be “contracted” to give a new graph G whose C^* -algebra $C^*(G)$ is Morita equivalent to $C^*(E)$. Crisp and Gow’s construction is widely applicable, as they point out in [10, §4]. It includes, for example, Morita equivalences of the C^* -algebras of graphs that are elementary-strong-shift-equivalent [4, 11], or are in- or out-delays of each other [5].

The C^* -algebra of a directed graph E is the universal C^* -algebra generated by mutually orthogonal projections p_v and partial isometries s_e associated to the vertices v and edges e of E , respectively, subject to relations. In particular, the relations capture the connectivity of the graph. For any subset V of vertices, $\sum_{v \in V} p_v$ converges to a projection p in the multiplier algebra of $C^*(E)$. (If V is finite, then p is in $C^*(E)$.) Then the module $pC^*(E)$ implements a Morita equivalence between the corner $pC^*(E)p$ of $C^*(E)$ and the ideal $C^*(E)pC^*(E)$ of $C^*(E)$. The difficult part is to identify $pC^*(E)p$ and $C^*(E)pC^*(E)$ with known algebras. The corner $pC^*(E)p$ may not be another graph algebra, but sometimes it is (see, for example, [9]). The projection p is called full when $C^*(E)pC^*(E) = C^*(E)$.

Now let R be a commutative ring with identity. A purely algebraic analogue of the graph C^* -algebra is the Leavitt path algebra $L_R(E)$ over R . This paper is based on the very simple observation that every subset V of the vertices of a directed graph E gives an algebraic version of the Morita equivalence between $pC^*(E)p$ and $C^*(E)pC^*(E)$ for Leavitt path algebras (see Theorem 1). We show that this observation is widely applicable by proving an algebraic version of Crisp and Gow’s theorem (see Theorem 3). A special case of this result has been very successfully used in both [2, Section 3] and [13].

If V is infinite, we cannot make sense of the projection p in $L_R(E)$, but we can make sense of the algebraic analogues of the sets $pC^*(E)$, $pC^*(E)p$ and $C^*(E)pC^*(E)$. For

Date: 12 January 2017.

2010 Mathematics Subject Classification. 16D70.

Key words and phrases. Directed graph, Leavitt path algebra, Morita context, Morita equivalence, graph algebra.

This research has been supported by a University of Otago Research Grant.

example,

$$pC^*(E) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \text{ are paths in } E \text{ and } \mu \text{ has range in } V\}$$

has analogue

$$M = \text{span}_R\{s_\mu s_\nu^* : \mu, \nu \text{ are paths in } E \text{ and } \mu \text{ has range in } V\},$$

where we also use s_e and p_v for universal generators of $L_R(E)$. Theorem 1 below gives a surjective Morita context (M, M^*, MM^*, M^*M) between the R -subalgebra MM^* and the ideal M^*M of $L_R(E)$. The set V is full, in the sense that $M^*M = L_R(E)$, if and only if the saturated hereditary closure of V is the whole vertex set of E (see Lemma 2).

Recently, the first author and Sims proved in [7, Theorem 5.1] that equivalent groupoids have Morita equivalent Steinberg R -algebras. They then proved that the graph groupoids of the graphs G and E appearing in Crisp and Gow's theorem are equivalent groupoids [7, Proposition 6.2]. Since the Steinberg algebra of a graph groupoid is canonically isomorphic to the Leavitt path algebra of the graph, they deduced that the Leavitt path algebras of $L_R(G)$ and $L_R(E)$ are Morita equivalent.

In particular, we obtain a direct proof of [7, Proposition 6.2] using only elementary methods. There are two advantages to our elementary approach: it illustrates on the one hand where we have had to use different techniques from the C^* -algebraic analogue, and on the other hand where we can just use the C^* -algebraic results already established.

2. PRELIMINARIES

A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 and E^1 , and range and source maps $r, s : E^1 \rightarrow E^0$. We think of E^0 as the set of vertices, and of E^1 as the set of edges directed by r and s . A vertex v is called a *infinite receiver* if $|r^{-1}(v)| = \infty$ and is called a *source* if $|r^{-1}(v)| = 0$. Sources and infinite receivers are called *singular* vertices.

We use the convention that a path is a sequence of edges $\mu = \mu_1\mu_2\cdots$ such that $s(\mu_i) = r(\mu_{i+1})$. We denote the i th edge in a path μ by μ_i . We say a path μ is finite if the sequence is finite and denote its length by $|\mu|$. Vertices are regarded as paths of length 0. We denote the set of finite paths by E^* and the set of infinite paths by E^∞ . We usually use the letters x, y for infinite paths. We extend the range map r to $\mu \in E^* \cup E^\infty$ by $r(\mu) = r(\mu_1)$; for $\mu \in E^*$ we also extend the source map s by $s(\mu) = s(\mu_{|\mu|})$.

Let $(E^1)^* := \{e^* : e \in E^1\}$ be a set of formal symbols called *ghost edges*. If $\mu \in E^*$, then we write μ^* for $\mu_{|\mu|}^* \cdots \mu_2^* \mu_1^*$ and call it a *ghost path*. We extend r and s to the ghost paths by $r(\mu^*) = s(\mu)$ and $s(\mu^*) = r(\mu)$.

Let R be a commutative ring with identity and let A be an R -algebra. A *Leavitt E -family in A* is a set $\{P_v, S_e, S_{e^*} : v \in E^0, e \in E^1\} \subset A$ where $\{P_v : v \in E^0\}$ is a set of mutually orthogonal idempotents, and

- (L1) $P_{r(e)}S_e = S_e = S_eP_{s(e)}$ and $P_{s(e)}S_{e^*} = S_{e^*} = S_{e^*}P_{r(e)}$ for $e \in E^1$;
- (L2) $S_{e^*}S_f = \delta_{e,f}P_{s(e)}$ for $e, f \in E^1$; and
- (L3) for all non-singular $v \in E^0$, $P_v = \sum_{r(e)=v} S_e S_{e^*}$.

For a path $\mu \in E^*$ we set $S_\mu := S_{\mu_1} \cdots S_{\mu_{|\mu|}}$. The *Leavitt path algebra $L_R(E)$* is the universal R -algebra generated by a universal Leavitt E -family $\{p_v, s_e, s_{e^*}\}$: that is, if A is an R -algebra and $\{P_v, S_e, S_{e^*}\}$ is a Leavitt E -family in A , then there exists a unique

R -algebra homomorphism $\pi : L_R(E) \rightarrow A$ such that $\pi(p_v) = P_v$ and $\pi(s_e) = S_e$ [15, §2-3]. It follows from (L2) that

$$L_R(E) = \text{span}_R\{s_\mu s_\nu^* : \mu, \nu \in E^*\}.$$

3. SUBSETS OF VERTICES OF A DIRECTED GRAPH GIVE MORITA EQUIVALENCES

Theorem 1. *Let E be a directed graph, let R be a commutative ring with identity and let $\{p_v, s_e, s_e^*\}$ be a universal generating Leavitt E -family in $L_R(E)$. Let $V \subset E^0$, and*

$$M := \text{span}_R\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu) \in V\} \text{ and } M^* := \text{span}_R\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\nu) \in V\}.$$

Then

- (1) MM^* is an R -subalgebra of $L_R(E)$;
- (2) $MM^* = \text{span}\{s_\mu s_\nu^* : r(\mu), r(\nu) \in V\}$, and M^*M is an ideal of $L_R(E)$ containing MM^* ;
- (3) with actions given by multiplication in $L_R(E)$, M is an MM^*-M^*M -bimodule and M^* is an M^*M-MM^* -bimodule;
- (4) there are bimodule homomorphisms

$$\Psi : M \otimes_{M^*M} M^* \rightarrow MM^* \quad \text{and} \quad \Phi : M^* \otimes_{MM^*} M \rightarrow M^*M$$

such that $(MM^, M^*M, M, M^*, \Psi, \Phi)$ is a surjective Morita context.*

Proof. We have

$$MM^* = \text{span}_R\{p_v s_\mu s_\nu^* s_\alpha s_\beta^* p_w : v, w \in V, \alpha, \beta, \mu, \nu \in E^*\}.$$

Products of the form $s_\mu s_\nu^* s_\alpha s_\beta^*$ are either zero or of the form $s_\mu s_\gamma s_\delta^* s_\nu^* = s_{\mu\gamma} s_{(\nu\delta)^*}$ for some $\gamma, \delta \in E^*$. Thus it is easy to see that MM^* is a subalgebra of $L_R(E)$ and

$$MM^* = \text{span}_R\{p_v s_\mu s_\nu^* p_w : v, w \in V, \mu, \nu \in E^*\} = \text{span}\{s_\mu s_\nu^* : \mu, \nu \in E^*, r(\mu), r(\nu) \in V\}.$$

Similarly, M^*M is an ideal.

To see that $MM^* \subset M^*M$, take a spanning element $s_\mu s_\nu^*$ of MM^* . Then $r(\mu) \in V$, $s_\mu s_\nu^* \in M$, and $s_\mu s_\nu^* = p_{r(\mu)} p_{r(\mu)^*} s_\mu s_\nu^* \in M^*M$. Thus $MM^* \subset M^*M$.

Since the module actions are given by multiplication in $L_R(E)$ it is easy to verify that M is an MM^*-M^*M -bimodule and M^* is an M^*M-MM^* -bimodule. The function $f : M \times M^* \rightarrow MM^*$ defined by $f(m, n) = mn$ is bilinear and $f(md, n) = f(m, dn)$ for all $d \in M^*M$. By the universal property of the balanced tensor product, there is a bimodule homomorphism $\Psi : M \otimes_{M^*M} M^* \rightarrow MM^*$ such that $\Psi(m \otimes n) = f(m, n) = mn$. Similarly, there is a bimodule homomorphism $\Phi : M^* \otimes_{MM^*} M \rightarrow M^*M$ such that $\Phi(n \otimes m) = nm$. Both Ψ and Φ are surjective. Since multiplication in $L_R(E)$ is associative, for $m, m' \in M, n, n' \in M^*$, we have

$$m\Phi(n \otimes m') = mn m' = \Psi(m \otimes n)m' \quad \text{and} \quad n\Psi(m \otimes n') = nm n' = \Phi(n \otimes m)n'.$$

Thus $(MM^*, M^*M, M, M^*, \Psi, \Phi)$ is a surjective Morita context. \square

In the situation of Theorem 1, we say a subset V of E^0 is *full* if the ideal M^*M is all of $L_R(E)$. We want a graph-theoretic characterisation of fullness, and so we want an algebraic version of [5, Lemma 2.2]. We need some definitions.

For $v, w \in E^0$ we write $v \leq w$ if there is a path $\mu \in E^*$ such that $s(\mu) = w$ and $r(\mu) = v$. We say a subset H of E^0 is *hereditary* if $v \in H$ and $v \leq w$ implies $w \in H$. A hereditary subset H of E^0 is *saturated* if

$$v \in E^0, 0 < |r^{-1}(v)| < \infty \text{ and } s(r^{-1}(v)) \subset H \implies v \in H.$$

We denote by $\Sigma H(V)$ the smallest saturated hereditary subset of E^0 containing V . For a saturated hereditary subset H of E^0 we write I_H for the ideal of $L_R(E)$ generated by $\{p_v : v \in H\}$.

Lemma 2. *Let E be a directed graph, and let $V \subset E^0$. Then V is full if and only if $\Sigma H(V) = E^0$.*

Proof. Let R be a commutative ring with identity and let $\{p_v, s_e, s_{e^*}\}$ be a universal generating Leavitt E -family in $L_R(E)$. As in Theorem 1, let $M = \text{span}_R\{s_\mu s_{\nu^*} : r(\mu) \in V\}$.

First suppose that V is full, that is, that $M^*M = L_R(E)$. To see that $\Sigma H(V) = E^0$, fix $v \in E^0$. Then $p_v \in M^*M$, and we can write p_v as a linear combination

$$p_v = \sum_{(\alpha, \beta) \in F_1, (\mu, \nu) \in F_2} r_{\alpha, \beta, \mu, \nu} s_\alpha s_{\beta^*} s_\mu s_{\nu^*}$$

where F_1, F_2 are finite subsets of $E^* \times E^*$, and each $r_{\alpha, \beta, \mu, \nu} \in R$ and $r(\beta) = r(\mu) \in V$.

Since $\Sigma H(V)$ is a hereditary subset containing V , we have $s(\beta), s(\mu) \in \Sigma H(V)$, and hence $p_{s(\beta)}, p_{s(\mu)} \in I_{\Sigma H(V)}$. Thus each summand

$$s_\alpha s_{\beta^*} s_\mu s_{\nu^*} = s_\alpha p_{s(\alpha)} s_{\beta^*} s_\mu p_{s(\mu)} s_{\nu^*} \in I_{\Sigma H(V)}.$$

It follows that $p_v \in I_{\Sigma H(V)}$. Thus $v \in \Sigma H(V)$, and hence $E^0 \subset \Sigma H(V)$. The reverse set inclusion is trivial. Thus $\Sigma H(V) = E^0$.

Conversely, suppose that $\Sigma H(V) = E^0$. To see that V is full, we need to show that the ideal M^*M is all of $L_R(E)$. For this, suppose that I is an ideal of $L_R(E)$ containing MM^* . It suffices to show that $L_R(E) = I$: by Theorem 1, M^*M is an ideal of $L_R(E)$ containing MM^* , and taking $I = M^*M$ gives $L_R(E) = M^*M$, as needed.

By [15, Lemma 7.6], the subset $H_I := \{v \in E^0 : p_v \in I\}$ of E^0 is a saturated hereditary subset of E^0 . Since I contains MM^* , we have $p_v \in I$ for all $v \in V$. Thus $V \subset H_I$, and since H_I is a saturated hereditary subset, we get $\Sigma H(V) \subset H_I$. By assumption, $\Sigma H(V) = E^0$, and now $L_R(E) = I_{E^0} = I_{\Sigma H(V)} \subset I_{H_I} \subset I \subset L_R(E)$. So $L_R(E) = I$ for any ideal I containing MM^* . Thus V is full. \square

4. CONTRACTIBLE SUBGRAPHS OF DIRECTED GRAPHS

We start by stating the algebraic version of Crisp and Gow's [10, Theorem 3.1]; for this we need a few more definitions.

Let E be a directed graph. A finite path $\alpha = \alpha_1 \alpha_2 \dots \alpha_{|\alpha|}$ in E with $|\alpha| \geq 1$ is a *cycle* if $s(\alpha) = r(\alpha)$ and $s(\alpha_i) \neq s(\alpha_j)$ when $i \neq j$. Then E (respectively, a subgraph) is *acyclic* if it contains no cycles. An acyclic infinite path $x = x_1 x_2 \dots$ in E is a *head* if each $r(x_i)$ receives only x_i and each $s(x_i)$ emits only x_i .

If E has a head, we can get a new graph F by collapsing the head down to a source. This is an example of a desingularisation, and hence $L_R(F)$ and $L_R(E)$ are Morita equivalent by [1, Proposition 5.2]. Thus the “no heads” hypothesis in Theorem 3 below is not restrictive.

Theorem 3. *Let R be a commutative ring with identity, let E be a directed graph with no heads, and let $\{p_v, s_e, s_{e^*}\}$ be a universal generating Leavitt E -family in $L_R(E)$. Suppose that $G^0 \subset E^0$ contains the singular vertices of E . Suppose also that the subgraph T of E defined by*

$$T^0 := E^0 \setminus G^0 \quad \text{and} \quad T^1 := \{e \in E^1 : s(e), r(e) \in T^0\}$$

is acyclic. Suppose that

(T1) each vertex in G^0 is the range of at most one infinite path $x \in E^\infty$ such that $s(x_i) \in T^0$ for all $i \geq 1$.

Also suppose that for each $y \in T^\infty$,

(T2) there is a path from $r(y)$ to a vertex in G^0 ;

(T3) $|s^{-1}(r(y_i))| = 1$ for all i ; and

(T4) $e \in E^1, s(e) = r(y) \implies |r^{-1}(r(e))| < \infty$.

Let G be the graph with vertex set G^0 and one edge e_β for each $\beta \in E^ \setminus E^0$ with $s(\beta), r(\beta) \in G^0$ and $s(\beta_i) \in T^0$ for $1 \leq i < |\beta|$, such that $s(e_\beta) = s(\beta)$ and $r(e_\beta) = r(\beta)$. Then $L_R(G)$ is Morita equivalent to $L_R(E)$.*

In words, the new graph G of Theorem 3 is obtained by replacing each path $\beta \in E^*$ with $s(\beta), r(\beta) \in G^0$ of length at least 1 which passes through T by a single edge e_β , which has the same source and range as β . Note that the edges e in E with $r(e)$ and $s(e)$ in G^0 remain unchanged.

Let $v \in E^0$. As in [10] define

$$B_v = \{\beta \in E^* \setminus E^0 : r(\beta) = v, s(\beta) \in G^0 \text{ and } s(\beta_i) \in T^0 \text{ for } 1 \leq i < |\beta|\}.$$

Then $\bigcup_{w \in G^0} B_w$ of E^* corresponds to the set of edges G^1 in G .

To prove Theorem 3, we apply Theorem 1 with $V = G^0$ so that

$$M = \text{span}_R\{s_\mu s_\nu^* : r(\mu) \in G^0\}.$$

Then M^*M is an ideal of $L_R(E)$ containing the subalgebra MM^* , and M^*M and MM^* are Morita equivalent. We need to show that $M^*M = L_R(E)$ and that MM^* is isomorphic to $L_R(G)$. Our proof uses quite a few of the arguments from Crisp and Gow's proof of [10, Theorem 3.1]. In particular, Lemma 3.6 of [10] gives a Cuntz-Krieger G -family in $C^*(E)$, and since the proof is purely algebraic it also gives a Leavitt G -family in $L_R(E)$. The universal property of $L_R(G)$ then gives a unique homomorphism $\phi: L_R(G) \rightarrow L_R(E)$. Crisp and Gow used the gauge-invariant uniqueness theorem to show that their C^* -homomorphism is one-to-one. The analogue here would be the graded uniqueness theorem, however ϕ is not graded. Instead, to show ϕ is one-to-one, we adapt some clever arguments from the proof of [1, Proposition 5.1] in Lemma 5 below which uses a reduction theorem.

Theorem 4 (Reduction Theorem). *Let R be a commutative ring with identity, let E be a directed graph, and let $\{p_v, s_e, s_{e^*}\}$ be a universal Leavitt E -family in $L_R(E)$. Suppose that $0 \neq x \in L_R(E)$. There exist $\mu, \nu \in E^*$ such that either:*

- (1) for some $v \in E^0$ and $0 \neq r \in R$ we have $0 \neq s_\mu^* x s_\nu = r p_v$, or*
- (2) there exist $m, n \in \mathbb{Z}$ with $m \leq n$, $r_i \in R$, and a non-trivial cycle $\alpha \in E^*$ such that $0 \neq s_\mu^* x s_\nu = \sum_{i=m}^n r_i s_\alpha^i$. (If i is negative, then $s_\alpha^i := s_{\alpha^*}^{|i|}$.)*

Proof. For Leavitt path algebras over a field this is proved in [3, Proposition 3.1]. We checked carefully that the same proof works over a commutative ring R with identity. \square

Lemma 5. *Let R be a commutative ring with identity. Let E and G be directed graphs, and let $\phi: L_R(G) \rightarrow L_R(E)$ be an R -algebra $*$ -homomorphism. Denote by $\{p_v, s_e, s_{e^*}\}$ and $\{q_v, t_e, t_{e^*}\}$ universal Leavitt E - and G -families in $L_R(E)$ and $L_R(G)$, respectively. Suppose that*

- (1) *for all $v \in G^0$, $\phi(q_v) = p_{v'}$ for some $v' \in E^0$; and*
- (2) *for all $e \in G^1$, $\phi(t_e) = s_\beta$ for some $\beta \in E^*$ with $|\beta| \geq 1$.*

Then ϕ is injective.

Proof. We follow an argument made in [1, Proposition 5.1]. Let $x \in \ker \phi$. Aiming for a contradiction, suppose that $x \neq 0$. By Theorem 4 there exist $\mu, \nu \in G^*$ such that either condition (1) or (2) of the theorem holds.

First suppose that (1) holds, that is, there exist $v \in G^0$ and $0 \neq r \in R$ such that $0 \neq t_\mu^* x t_\nu = r q_v$. Using assumption (1), there exists $v' \in E^0$ such that $\phi(q_v) = p_{v'}$. Now

$$0 = \phi(t_\mu^* x t_\nu) = \phi(r q_v) = r \phi(q_v) = r p_{v'}.$$

But $r p_{v'} \neq 0$ for all $0 \neq r \in R$, giving a contradiction.

Second suppose that (2) holds, that is, there exist $m, n \in \mathbb{Z}$ with $m \leq n$, $r_i \in R$, and a non-trivial cycle $\alpha \in E^*$ such that $0 \neq s_\mu^* x s_\nu = \sum_{i=m}^n r_i s_\alpha^i$. Since α is a non-trivial cycle, it has length at least 1. By assumption (2), $\phi(t_\alpha) = s_{\alpha'}$ where α' is a path in E such that $|\alpha'|_E \geq |\alpha|_G \geq 1$. Since ϕ is an R -algebra $*$ -homomorphism we get

$$0 = \phi(t_\mu^* x t_\nu) = \phi\left(\sum_{i=m}^n r_i t_\alpha^i\right) = \sum_{i=m}^n r_i \phi(t_\alpha)^i = \sum_{i=m}^n r_i s_{\alpha'}^i.$$

Since $|\alpha'| = k$ for some $k \geq 1$, $s_{\alpha'}$ has grading k , and hence each $s_{\alpha'}^i$ has grading ik . Thus each term in the sum $\sum_{i=m}^n r_i s_{\alpha'}^i$ is in a distinct graded component. But since $s_{\alpha'} \neq 0$, we must have $r_i = 0$ for all i . Thus $\sum_{i=m}^n r_i t_\alpha^i = 0$, which is a contradiction. In either case, we obtained a contradiction to the assumption that $x \neq 0$. Thus $x = 0$ and ϕ is injective. \square

Proof of Theorem 3. Let $\{p_v, s_e, s_{e^*}\}$ be a universal Leavitt E -family in $L_R(E)$. We apply Theorem 1 with $V = G^0$ to get a surjective Morita context between MM^* and M^*M .

Since M and $M^* \subset L_R(E)$, we have $M^*M \subset L_R(E)$. To see that $L_R(E) \subset M^*M$, let $s_\mu s_\nu^* \in L_R(E)$. We may assume that $s(\mu) = s(\nu)$, for otherwise $s_\mu s_\nu^* = 0$. If $s(\mu) \in G^0$, then the Leavitt E -family relations give $s_\mu s_\nu^* = s_\mu s_{s(\mu)^*} s_{s(\mu)} s_\nu^* \in M^*M$ and we are done. So suppose $s(\mu) \in T^0$. Then the graph-theoretic [10, Lemma 3.4(c)] implies that $B_{s(\mu)} \neq \emptyset$. Suppose first that $B_{s(\mu)}$ is finite. It then follows from the first part of [10, Lemma 3.6] that $s(\mu)$ is a nonsingular vertex. The second part of [10, Lemma 3.6] implies that for any Cuntz-Krieger E -family $\{P_v, S_e, S_{e^*}\}$ in $C^*(E)$,

$$P_{s(\mu)} = \sum_{\beta \in B_{s(\mu)}} S_\beta S_{\beta^*};$$

the proof is purely algebraic and works for any Leavitt E -family in $L_R(E)$. Thus

$$s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* = \sum_{\beta \in B_{s(\mu)}} s_\mu S_\beta S_{\beta^*} s_\nu^* = \sum_{\beta \in B_{s(\mu)}} s_{\mu\beta} s_{s(\beta)^*} s_{s(\beta)} s_{(\nu\beta)^*} \in M^*M.$$

Next suppose that $B_{s(\mu)}$ is infinite. Since $s(\mu) \in T^0$ and $B_{s(\mu)}$ is infinite, the graph-theoretic [10, Lemma 3.4(d)] implies that there exists $x \in T^\infty$ such that $s(\mu) = r(x)$. By

assumption (T2), there is a path $\alpha \in E^*$ with $r(\alpha) \in G^0$ such that $s(\alpha) = r(x) = s(\mu)$. Now

$$s_\mu s_{\nu^*} = s_\mu p_{s(\mu)} s_{\nu^*} = s_\mu s_{\alpha^*} s_{\alpha} s_{\nu^*} \in M^* M.$$

Thus $L_R(E) = M^* M$. (We could have used Lemma 2 to prove that $L_R(E) = M^* M$, as Crisp and Gow do, but this seemed easier.)

Next we show that $L_R(G)$ and $M^* M$ are isomorphic. For $v \in G^0$ and $\beta \in \cup_{w \in G^0} B_w$ define

$$Q_v = p_v, \quad T_{e_\beta} = s_\beta \quad \text{and} \quad T_{e_\beta^*} = s_{\beta^*}.$$

Then $\{Q_v, T_e, T_{e^*}\}$ is a Leavitt G -family in $L_R(E)$; again this follows as in the proof of [10, Theorem 3.1]. To see what is involved, we briefly step through this. Relations (L1) follow immediately from the relations for $\{p_v, s_e, s_{e^*}\}$. To see that (L2) holds, let $\gamma, \beta \in \cup_{w \in G^0} B_w$. Then $T_{e_\beta^*} T_{e_\gamma} = s_{\beta^*} s_\gamma$. By the graph-theoretic [10, Lemma 3.4(a)] neither γ nor β can be a proper extension of the other. Thus $T_{e_\beta^*} T_{e_\gamma} = s_{\beta^*} s_\gamma = \delta_{\beta, \gamma} p_{s(\beta)} = \delta_{e_\beta, e_\gamma} Q_{s(e_\beta)}$, and (L2) holds.

To see that (L3) holds, let $v \in G^0$ be a non-singular vertex. Then B_v is finite and non-empty because it is equinumerous with $r_G^{-1}(v)$. Using the algebraic analogue of [10, Lemma 3.6] again, we have

$$Q_v = p_v = \sum_{\beta \in B_v} s_\beta s_{\beta^*} = \sum_{e_\beta \in r_G^{-1}(v)} T_{e_\beta} T_{e_\beta^*}.$$

Thus (L3) holds and $\{Q_v, T_e, T_{e^*}\}$ is a Leavitt G -family in $L_R(E)$.

Now let $\{q_v, t_e, t_{e^*}\}$ be a universal Leavitt G -family in $L_R(G)$. The universal property of $L_R(G)$ gives a unique homomorphism $\phi: L_R(G) \rightarrow L_R(E)$ such that for $v \in G^0$, $\beta \in \cup_{w \in G^0} B_w$,

$$\phi(q_v) = Q_v = p_v, \quad \phi(t_{e_\beta}) = T_{e_\beta} = s_\beta \quad \text{and} \quad \phi(t_{e_\beta^*}) = T_{e_\beta^*} = s_{\beta^*}.$$

If $v \in G^0$, then $p_v = s_v s_{v^*} \in MM^*$; if $\beta \in B_w$ for some $w \in G^0$, then $r(\beta) \in G^0$, and $s_\beta = s_\beta s_{s(\beta)^*}$ and $s_{\beta^*} = s_{s(\beta)} s_{\beta^*} \in MM^*$. It follows that the range of ϕ is contained in MM^* . That ϕ is onto MM^* again follows from work of Crisp and Gow. They take a non-zero spanning element $s_\mu s_{\nu^*} \in MM^*$ and use the graph-theoretic [10, Lemma 3.4(b)], the algebraic [10, Lemma 3.6] and assumptions (T1)-(T4) to show that $s_\mu s_{\nu^*}$ is in the range of ϕ . Thus ϕ is onto.

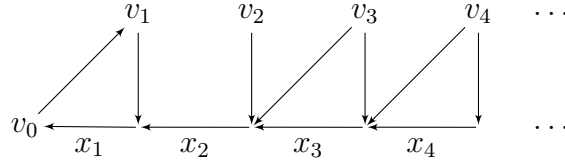
Finally, ϕ satisfies the hypotheses of Lemma 5, and hence is one-to-one. Thus ϕ is an isomorphism of $L_R(G)$ onto MM^* . \square

Remark 6. A version of Theorem 1 should hold for the Kumjian-Pask algebras associated to locally convex or finitely aligned k -graphs [6, 8]. But the challenge would be to formulate an appropriate notion of contractible subgraph in that setting.

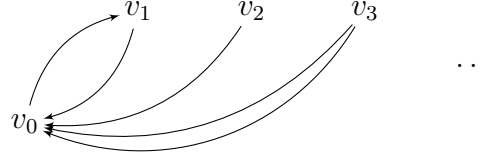
5. EXAMPLES

As mentioned in the introduction, the setting of Theorem 3 includes many known examples. We found it helpful to see how some concrete examples fit.

Example 7. An infinite path $x = x_1 x_2 \dots$ in a directed graph is *collapsible* if x has no exits except at $r(x)$, the set $r^{-1}(r(x_i))$ of edges is finite for every i , and $r^{-1}(r(x)) = \{x_1\}$ (see [14, Chapter 5]). Consider the following row-finite directed graph E :

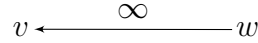


The infinite path $x = x_1 x_2 \dots$ is collapsible. When we collapse x to the vertex v_0 , as described in [14, Proposition 5.2], we get the following graph F with infinite receiver v_0 :

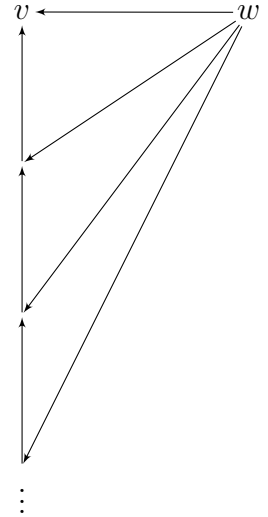
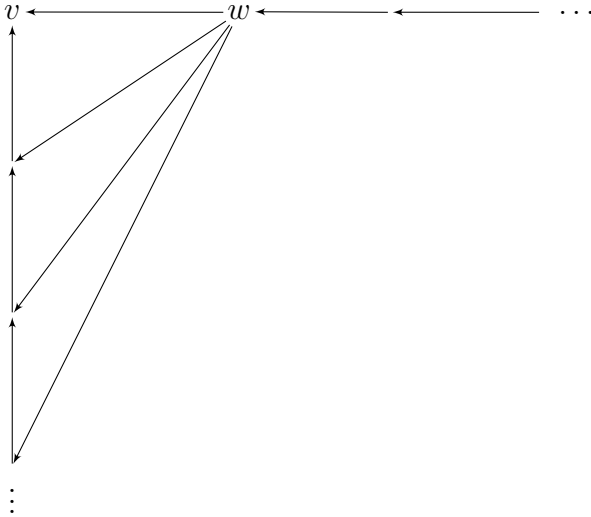


This fits the setting of Theorem 3: take $G^0 = \{v_i : i \geq 0\}$, and then T is the subgraph defined by $T^0 = \{s(x_i) : i \geq 1\}$ and $T^1 = \{x_i : i \geq 2\}$. Then T contains none of the singular vertices $\{v_i : i \geq 2\}$ of E , is acyclic, and satisfies the conditions (T1)–(T4). Thus F is the graph G described in the theorem.

Example 8. Consider the directed graph F with source w and infinite receiver v :



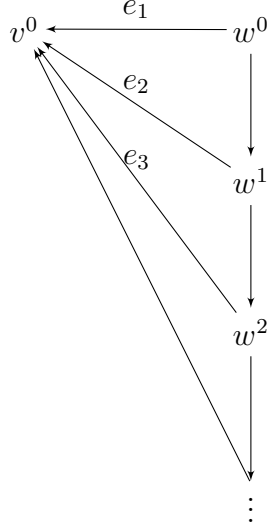
An example of a Drinen-Tomforde desingularisation [12] of F is the row-finite graph E with no sources below on the left: a head has been added at the source w of F and each edge from w to v in F has been replaced with paths as shown. (This desingularisation is also an example of an out-delay.) Then, since we are interested in Morita equivalence, we delete the head at w to get the graph E below on the right.



Set $T^0 = E^0 \setminus \{v, w\}$ and $T^1 = \{e \in E^1 : s(e), r(e) \in T^0\}$. Then the subgraph T contains none of the singularities of E , is acyclic, and satisfies conditions (T1)–(T4) of Theorem 3. The graph F we started with is the graph G of Theorem 3.

Example 9. Consider again the graph F of Example 8. Label the infinitely many edges from w to v by e_i for $i \geq 1$. This time we will consider the in-delayed graph $d_s(E)$ given

by the Drinen source-vector $d_s: E^0 \cup E^1 \rightarrow \mathbb{N} \cup \{\infty\}$ (see [5, Section 4]) to be the function defined by $d_s(e_i) = i - 1$ for $i \geq 1$, $d_s(v) = 0$ and $d_s(w) = \infty$. Then the in-delayed graph $d_s(E)$ given by d_s , as described in [5], is



Now take $T^0 = d_s(E)^0 \setminus \{v^0, w^0\}$. Then T^0 contains none of the singular vertices of $d_s(E)$, and the corresponding subgraph T is acyclic. There are no infinite paths in $d_s(E)$, and hence conditions (T1)–(T4) of Theorem 3 hold trivially. The graph G of the theorem is again the graph F that we started out with.

REFERENCES

- [1] G. Abrams and G. Aranda Pino. The Leavitt path algebras of arbitrary graphs. *Houston J. Math. Univ.* **34** (2008), 423–442.
- [2] G. Abrams, A. Louly, E. Pardo, and C. Smith. Flow invariants in the classification of Leavitt path algebras. *J. Algebra* **333** (2011), 202–231.
- [3] G. Aranda Pino, D. Martín Barquero, C. Martín González and M. Siles Molina. The socle of a Leavitt path algebra. *J. Pure Appl. Algebra* **212** (2008), 500–509.
- [4] T. Bates. Applications of the gauge-invariant uniqueness theorem for graph algebras. *Bull. Austral. Math. Soc.* **66** (2002), 57–67.
- [5] T. Bates and D. Pask. Flow equivalence of graph algebras. *Ergodic Theory Dynam. Sys.* **24** (2004), 367–382.
- [6] L.O. Clark, C. Flynn and A. an Huef. Kumjian-Pask algebras of locally convex higher-rank graphs. *J. Algebra* **399** (2014), 445–474.
- [7] L.O. Clark and A. Sims. Equivalent groupoids have Morita equivalent Steinberg algebras. *J. Pure Appl. Algebra* **219** (2015), 2062–2075.
- [8] L.O. Clark and Y.E.P. Pangalela. Kumjian-Pask algebras of finitely-aligned higher-rank graphs. arXiv:1512.06547.
- [9] T. Crisp. Corners of graph algebras. *J. Operator Theory* **60** (2008), 253–271.
- [10] T. Crisp and D. Gow. Contractible subgraphs and Morita equivalence of graph C^* -algebras. *Proc. Amer. Math. Soc.* **134** (2006), 2003–2013.
- [11] D. Drinen and N. Sieben. C^* -equivalences of graphs. *J. Operator Theory* **45** (2001), 209–229.
- [12] D. Drinen and M. Tomforde. The C^* -algebras of arbitrary graphs. *Rocky Mountain J. Math.* **35** (2005), 105–135.
- [13] R. Johansen and A.P.W. Sørensen. The Cuntz splice does not preserve $*$ -isomorphism of Leavitt path algebras over \mathbb{Z} . *J. Pure Appl. Algebra* **220** (2016), 3966–3983.
- [14] I. Raeburn, *Graph Algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, Amer. Math. Soc., Providence, 2005.

- [15] M. Tomforde. Leavitt path algebras with coefficients in a commutative ring. *J. Pure Appl. Algebra* **215** (2011), 471–484.

(L.O. Clark, A. an Huef, P. Luiten-Apirana) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTAGO, P.O. BOX 56, DUNEDIN 9054, NEW ZEALAND

E-mail address: lclark, astrid @maths.otago.ac.nz

E-mail address: pareorangeluitenapirana@gmail.com